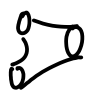
J. Lurie – 26/5/08 – TQFTs and string topology

- <u>d Cob</u> : category with objects = (d-1)-mflds
  morphisms = bordisms up to diffeomorphism

  <u>Note</u>: all manifolds are smooth, compact, oriented

  <u>Def</u>: (Atiyah) ‖ A (d-dim!) TQFT is a $\otimes$-functor
  $F: d\,Cob \to \mathbb{C}$ vector spaces, ie. $F(M \amalg N) \cong F(M) \otimes F(N)$
  $F(\emptyset) \cong \mathbb{C}$

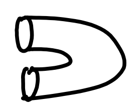
- <u>Ex</u>: $d=2$: $F(S^1) = A$ vector space
  $\left(\rightsquigarrow F\left(\circ\ \circ\ \circ\right) = A^{\otimes 3}\right)$

  F() : $A \otimes A \xrightarrow{m} A$

  m makes A a comm. assoc. unital algebra over $\mathbb{C}$

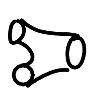
  F() : $A \xrightarrow{tr} \mathbb{C}$

  tr is nondegenerate: $A \otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$ nondeg.

  $\rightsquigarrow$ A is a <u>Frobenius algebra</u>

- Note: M closed d-manifold $\rightsquigarrow$ $F(M): \mathbb{C} \to \mathbb{C}$ multiplication by a complex number $F(M)$.

- In low dim., above definition allows us to chip M into small pieces and compute from data for S^1,  ,  ....

  In higher dim. we'd like to chip M into smaller pieces, allowing corners etc. $\to$ <u>extended TFT's</u> (higher categories)

Sketch of definition:

An extended TQFT of dim. d is a rule

- closed d-manifold $\rightsquigarrow$ complex number
- closed $(d-1)$-manifold $\rightsquigarrow$ $\mathbb{C}$-vector space
- bordism of $(d-1)$-mfds $\rightsquigarrow$ $\mathbb{C}$-linear map
- closed $(d-2)$-manifold $\rightsquigarrow$ $\mathbb{C}$-linear category
- bordism of $(d-2)$-mflds $\rightsquigarrow$ ($\mathbb{C}$-linear) functor

satisfying reasonable axioms, ie.

$\|$ an extended TQFT is a $\otimes$-functor between n-categories

It looks like this is a lot of data, but lots of structure

$\Rightarrow$ things should be simpler!

Namely: ext$^{\text{d}}$ TQFT structure allows us to cut mflds into very simple std pieces, and the data for these should determine everything.

$\rightsquigarrow$ Baez-Dolan cobordism hypothesis:

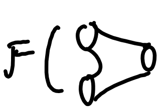
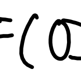
Extended TQFTs are "easy to describe/construct".

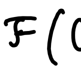
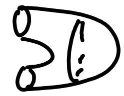
Non-example: string topology:

$d=2$, M a manifold of any dim., $LM = Map(S^1, M)$

There almost exists a TQFT $\mathcal{F}$ st. $\mathcal{F}($  $) = H_*(LM)$

| F is functorial only for bordisms Σ st. each component has non-empty outgoing boundary.

$\mathcal{F}($  $)$ = string topology product, but $\nexists$ $\mathcal{F}($  $) = tr$

(Indeed: if $\mathcal{F}($$) = tr$ and  $\Rightarrow$ $tr \circ m$ is a perfect pairing on $\mathcal{F}(S^1)$, would imply $\dim \mathcal{F}(S^1) < \infty$).

<u>Q</u>: does this example come from a TQFT valued in chain complexes?

<u>A</u>: yes, if things are properly formulated.

<u>Chain complex valued TQFTs:</u>

- $\mathcal{F}(M^{d-1})$ = some chain complex
- $\mathcal{F}(M \boxed{B} N)$ = chain map $\mathcal{F}(M)$ to $\mathcal{F}(N)$

Our old axioms would require that, if B & B' are diffeomorphic, then they induce the same chain map; this isn't realistic. Instead, ask that B & B' induce <u>homotopic</u> chain maps, where the homotopy depends on the diffeomorphism.

$\rightarrow$ <u>instead:</u> $\mathrm{Bord}(M,N)$:= classifying space ("moduli space") of bordisms from M to N

$\Rightarrow$ we see a $\boxed{\text{chain map } \psi : C_*(\mathrm{Bord}(M,N)) \longrightarrow \mathrm{Hom}(F(M), F(N))}$

(<u>ie</u>: a bordism gives a chain map $F(M) \to F(N)$
a homotopy of bordisms  gives a homotopy b/w chain maps; ...)

$\rightsquigarrow$ <u>Reformulation:</u>
$$\begin{cases} C_*(\mathrm{Bord}(M,N)) \otimes \mathcal{F}(M) \to \mathcal{F}(N). \\ H_*(\mathrm{Bord}(M,N)) \otimes H_*\mathcal{F}(M) \to H_*\mathcal{F}(N). \end{cases}$$

[<u>e.g.</u>: for Riem. surfaces, classif space ~ mapping class groups, and this gives actions of $H_*(\text{M.C.G.})$ on various things....]

Reformulation #2:

Up to homotopy, ψ is equivalent to giving $\mathrm{Bord}(M,N) \to \mathrm{Map}(\mathcal{F}(M), \mathcal{F}(N))$

topological space s.t. $\pi_i \,\mathrm{Map} \approx$ chain homotopy classes of degree i.

ie: a chain-complex-valued TQFT is a $\otimes$-functor between topological categories

means: higher category theory s.t. n-morphisms are invertible for $n>1$.

(Sketch) def^n: 1-Bord: symmetric monoidal $(\infty,1)$-category wrt $M \otimes N := M \amalg N$

objects $=$ 0-manifolds

morphisms: $\mathrm{Hom}_{1\mathrm{Bord}}(M,N) = \mathrm{Bord}(M,N)$

$\Rightarrow$ Goal: understand $\otimes$-functors $F: 1\,\mathrm{Bord} \to \mathcal{C}$.

$F: 1\,\mathrm{Bord} \to \mathcal{C} \quad \Rightarrow$

$F(\emptyset) = 1 \in \mathcal{C}$

$F(\bullet^{+}) = X \in \mathcal{C}$

$F(\bullet^{-}) = Y \in \mathcal{C}$

$F(\subset) = 1 \to X \otimes Y$

$F(\supset) := X \otimes Y \to 1$

These exhibits X & Y as duals.

$\Rightarrow X$ is dualizable, & can forget Y.

Thm: $\mathrm{Fun}^{\otimes}(1\text{-}\mathrm{Bord}, \mathcal{C}) \xrightarrow[\simeq]{ev(\bullet)} \{\text{dualizable objects of } \mathcal{C}\}$

Key point:

Need to understand $\mathcal{F}(\circlearrowleft) = \mathcal{F}(\subset \supset)$

$$1 \xrightarrow[\text{coev}]{} X \otimes X^\vee \xrightarrow[\text{ev}]{} 1.$$

$\equiv \dim X \in \mathrm{Hom}_{\mathcal{C}}(1,1)$

In fact, want

$$\begin{array}{ccc} \mathrm{Hom}_{1\mathrm{Bord}}(\emptyset, \emptyset) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(1,1) \\ \cup\!\!| & & \cup\!\!| \\ B\mathrm{Diff}^+(S^1) & & \dim X \\ \wr\!\wr & \nearrow & \\ \mathbb{CP}^\infty \ni * & \mapsto & \end{array}$$

we've only described this on one point (broke symmetry of S^1).

... think of $B\mathrm{Diff}^+(S^1)$ as a point with S^1-action on it

$\Rightarrow$ image = a point w/ S^1-action.

We've defined the point ($= \dim X$), need to check the S^1-action as well.

Claim: it all works out & we get the theorem above. $\triangle$

Real goal: understand $\otimes$-functors $2\text{-Bord} \longrightarrow \mathcal{C}$ $(\infty,2)$-category

objects: 0-manifolds

morphisms: bordisms of 0-mflds

higher morphisms: $\mathrm{Bord}(M,N)$ spaces

by above thm,

$$\boxed{\begin{array}{l} \mathcal{F}(\bullet^+) = X \in \mathcal{C} \text{ dualizable} \\ \mathcal{F}(\circlearrowleft) = \dim X \in \mathrm{Hom}_{\mathcal{C}}(1,1), \text{ has } \underline{S^1\text{-action}} \\ \mathcal{F}(\bigcirc) = \eta \in 2\mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_1, \dim X)^{S^1} \end{array}}$$

• this is all that's needed to determine the functor!

Thm: $$\mathrm{Fun}^{\otimes}(2\mathrm{Bord}, \mathcal{C}) \simeq \left\{ \begin{array}{l} X \in \mathcal{C} \text{ dualizable} \\ \eta \in 2\mathrm{Hom}(id_1, \dim X)^{S^1} \text{ nondegenerate} \end{array} \right\}$$

<u>Example</u> (<u>string topology</u>)

k a field $\rightsquigarrow$ $\mathcal{C}$ = "dg-categories / k"

objects = dg-categories

morphisms = "dg-functors"

2-morphisms = "nat. transformations"

M a manifold $\rightsquigarrow$ $X \in \mathcal{C}$:

• $$\underline{X := \left\{ \begin{array}{l} \text{chain complexes of sheaves on } M \\ \text{with loc. constant cohomology} \end{array} \right\}}$$

X dualizable $\rightsquigarrow$ X has a dimension

• $\dim X \in \hom_{\mathcal{C}}(1,1)$ = dg-functors dg-Vect ↺

$\cong$ chain complexes of dg-vect-spaces

Here $\underline{\dim X \cong C_*(LM, k)}$. ; S^1-action = action on loop space ✓

• η should be an <u>S^1-invariant cycle in $C_*(LM;k)$</u>

To get η, start with a cycle in $C_*(\underbrace{LM^{S^1}}_{=\ \underline{\text{const. loops}}})$: !

$$C_*(M) \simeq C_*(LM^{S^1}) \xrightarrow[\text{incl. of constant loops}]{} C_*(LM)^{S^1}$$

$\Downarrow$

$[M]$ ← this is our choice

nondegeneracy $\Leftrightarrow$ Poincaré duality

<u>Corollary</u>: string topology operations are well-defined for any PD space, and are homotopy invariants.

<u>Rmk</u>: in fact, we said above that string topology isn't quite a TQFT because need non-empty outgoing ∂

$\Rightarrow$ in fact it's a functor from a subcategory

$$2\mathrm{Bord}^0 \to \mathcal{C}$$

require non-empty outgoing ...

but to the level of detail given above, classif^{n} theorem still holds

[$2\mathrm{Bord}^0 \to \mathcal{C}$ need nondegeneracy cond. on η as above
$2\mathrm{Bord} \to \mathcal{C}$ need stronger nondegeneracy cond^{n}, can't hold in our case.]